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A passivity-based approach to reset control systems stability[☆]

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ABSTRACT

The stability of reset control systems has been mainly studied for the feedback interconnection of reset compensators with linear time-invariant systems. This work gives a stability analysis of reset compensators in feedback interconnection with passive nonlinear systems. The results are based on the passivity approach to $\mathcal{L}_2$-stability for feedback systems with exogenous inputs, and the fact that a reset compensator will be passive if its base compensator is passive. Several examples of full and partial reset compensations are analyzed, and a detailed case study of an in-line pH control system is given.

1. Introduction

Reset control system design was initiated fifty years ago with the work of Clegg [1], who introduced a nonlinear integral feedback controller based on a reset action of the integrator, the so-called *Clegg integrator*. The reset action amounts to setting the integrator output equal to zero whenever its input is zero. In this way a faster system response without excessive overshoot may be expected, thus possibly overcoming a basic limitation of the standard linear integral feedback. This has spurred the development of several other nonlinear compensators, all based on describing function analysis. Furthermore, in a series of papers by Horowitz and coworkers [2,3] reset control systems have been advanced by introducing the first-order reset element (FORE).

One of the main drawbacks of reset compensators is that the stability of the feedback system is not always guaranteed by the stability of the underlying linear time-invariant (LTI) system without reset action. In fact it is well known, and easily illustrated, that the reset action can destabilize a stable LTI feedback system. Recently, the problem for linear reset control systems has been successfully addressed in [4,5] for general reset compensators, allowing full or partial state reset. As a result stability of the reset control system can be checked by the (strictly) positive realness of a certain transfer matrix H_β, referred to as the H_β-condition.

Reset control systems can be also regarded as a special case of *hybrid systems*, or as systems with impulsive motion. From this perspective, the recent work [6] addressed the stability problem of these types of systems with the goal of analyzing the stability of switching between LTI controllers. Furthermore, the H_β-condition has been relaxed in [7] to obtain a less restrictive Lyapunov stability condition. The papers [8,9] derive conditions based on the reset times that can be used both for stable and unstable linear systems.

Regarding the $\mathcal{L}_2$-stability of reset systems with inputs, a number of papers have appeared that give results for particular cases of reset compensators and/or inputs. The work [7] approaches the problem for compensators in which its output has the same sign as its input, and the zero reference case is considered. In addition, in [10] $\mathcal{L}_2$-stability conditions for the case of nonzero references are given. The conservatism given by H_β-condition is improved for these kinds of systems.

On the other hand, dissipative systems theory was developed in [11], where the concept of a *passive system*, originating from electric circuit theory and mechanical systems, was extended to abstract systems. A main theorem in this context is the fact that the feedback interconnection of two passive nonlinear systems is again a passive system. Passivity techniques have been shown to be a powerful tool for nonlinear control, see e.g. [12]. Dissipative systems theory has been developed for hybrid systems in [13], where notions such as *supply rate* have been extended to the hybrid case. We also refer to [14–17] for work on passivity of

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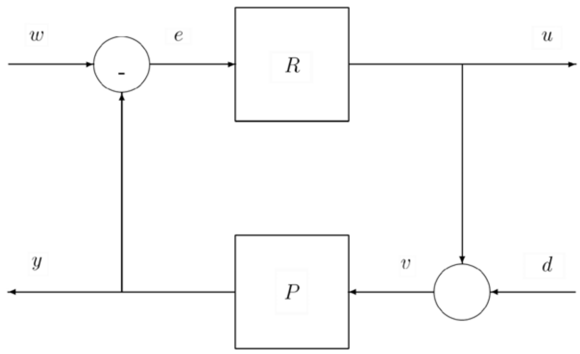

Fig. 1. Reset controller R applied to a LTI plant.

hybrid systems. In spite of the fact that a single-input single-output approach to passive systems theory will not be sufficient for more general hybrid systems, this paper will show how several passivity properties can be obtained for reset compensators. In [13], this kind of impulsive systems are referred to as *input-dependent impulsive dynamical systems*.

The goals of this work are: (a) to obtain stability conditions that are applicable to feedback interconnections of linear compensators with reset action and nonlinear plants; (b) to find passive reset compensators that can be used in passive control techniques. Passivity conditions for stability will be developed, which are easily checked on the linear compensator *without* reset action.

The structure of the work is the following. In Section 2, a description of the problem setup is given and some basic results about passivity theory are recalled. Section 3 gives the main results about the passivity properties of reset compensator, which are used to show $\mathcal{L}_2$-stability with respect to reference and perturbation inputs. In Section 4, an application to an industrial nonlinear plant is developed.

2. Preliminaries and problem setup

This work approaches the stability problem of reset control systems with inputs using general passivity theory. We consider the feedback system given by Fig. 1, where w and d are the reference and disturbance inputs, respectively. R is a single-input single-output (SISO) reset compensator to be defined later on, and P is a single-input single-output (SISO) plant. The set $\mathcal{L}_2$ consists of all measurable functions $f(\cdot) : \mathbb{R}_+ \to \mathbb{R}$ such that $\int_0^\infty |f(t)|^2 \mathrm{d}t < \infty$, being the $\mathcal{L}_2$-norm $\|\cdot\| : \mathcal{L}_2 \to \mathbb{R}_+$ defined by $\|f\| = \left(\int_0^\infty |f(t)|^2 \mathrm{d}t\right)^{\frac{1}{2}}$.

The feedback interconnection in system (Fig. 1) is given simply by

$$e(t) = w(t) - y(t), \qquad u(t) = v(t) + d(t). \tag{1}$$

The feedback system of Fig. 1 is called $\mathcal{L}_2$-stable (with respect to inputs w and d) if for every input signals $w \in \mathcal{L}_2$ and $d \in \mathcal{L}_2$ the outputs $u \in \mathcal{L}_2$ and $y \in \mathcal{L}_2$. In addition, it is finite-gain stable if there exists a positive constant $\gamma > 0$ such that $\|y\|^2 + \|u\|^2 \le \gamma(\|w\|^2 + \|d\|^2)$.

The plant P is represented by the state-space model

$$P : \begin{cases} \dot{x}_p = f(x_p, u), & u \in \mathbb{R} \\ y = g(x_p, u), & y \in \mathbb{R} \end{cases} \tag{2}$$

where n_p is the dimension of the state x_p, $f : \mathbb{R}^{n_p} \times \mathbb{R} \to \mathbb{R}^{n_p}$ is locally Lipschitz, $g : \mathbb{R}^{n_p} \times \mathbb{R} \to \mathbb{R}$ is continuous, $f(0,0) = 0$, and $g(0,0) = 0$. In addition, following the framework given in [13], the dynamics of the reset compensator is described by three elements: (a) a continuous-time dynamical equation, (b) a difference equation, and (c) a reset law. During time intervals in which the reset law is not applied, the system evolves in a continuous fashion; otherwise, when the resetting law is applied, the system undergoes a *jump*. We will throughout consider reset compensators R that consist of an LTI compensator (the so-called *base linear compensator*) together with a reset action, given by the following impulsive differential equation *(IDE)*:

$$R : \begin{cases} \dot{x}_r = A_r x_r + B_r e, & e \neq 0 \\ x_r^+ = A_\rho x_r, & e = 0 \\ u = C_r x_r + D_r e \end{cases} \tag{3}$$

where n_r is the dimension of the state x_r, A_ρ is a diagonal matrix with diagonal elements equal to zero for the state components to be reset, and equal to one for the rest of the compensator states, n_ρ is defined as the dimension of the reset subspace, and $n_{\overline{\rho}}$ is defined as the dimension of the non-reset subspace ($n_\rho + n_{\overline{\rho}} = n_r$). When $A_\rho = 0$, R will be referred to as *full reset compensator*; otherwise, it will be referred to as *partial reset compensator*.

The first equation in (3) describes the continuous compensator dynamics at the non-reset time instants, while the second equation gives the reset operation as a jump of the compensator state at the reset instants. Note that reset time instants occur when the compensator input is zero. The base compensator is simply obtained by omitting the reset actions in (3), and thus has the transfer function $R_{bl}(s) = C_r(sI - A_r)^{-1}B_r + D_r$. Henceforth, whenever a transfer function is used to describe a reset compensator, this means that its base linear compensator has this transfer function. Furthermore, we will use the notations x_r^+ or $x_r(t^+)$ for the value $x_r(t+\tau)$ with $\tau \to 0^+$.

Impulsive systems such as (3) are a special case of hybrid systems, for which it is well that phenomena like Zeno behavior and beating may occur [13]. To avoid these phenomena, we will assume throughout this paper that the solutions to (3) are *time regularized* (see for example [7] and the references therein), which means that the reset law is switched off for a time interval of length $\Delta_m > 0$ after each reset time. Thus formally speaking we consider the following reset system

$$R : \begin{cases} \dot{\Delta} = 1, & \dot{x}_r = A_r x_r + B_r e, \quad e \neq 0 \text{ or } \Delta < \Delta_m \\ \Delta^+ = 0, & x_r^+ = A_\rho x_r, \quad e = 0 \text{ and } \Delta \geq \Delta_m \\ u = C_r x_r + D_r e \end{cases} \tag{4}$$

with zero initial conditions: $\Delta(0) = 0$, $x_r(0) = 0$. As a consequence of time regularization there will exist for any input e only a *finite* number of reset times on any finite time interval, hence excluding Zeno behavior. Furthermore, on the infinite time interval $[t_0, \infty)$ there will exists a countable set $\{t_1, t_2, \ldots, t_k, \ldots\}$ where $t_{k+1} - t_k \geq \Delta_m$ for all $k = 1, 2, \ldots$, and in addition the constant Δ_m does not depend on the input e.

In contrast to the works [7] and [18], where the reset action is active when input and output have a different sign, the original definition of reset according to [1,3,2,5] has been used here. Note that the definition of Clegg integrator proposed in [18] is equivalent to the original in [1] in the case of zero initial condition. In addition, although the definition given in [18] has advantages in some particular cases, it cannot be applied to partial reset systems. This is the main reason why the original definition has been used in this work.

A system $H : \mathcal{L}_{2,e} \to \mathcal{L}_{2,e}$, with input u and output $y = Hu$ is said to be *passive* if there exists a constant $\beta \leq 0$ such that

$$\int_0^T u^\top(t) y(t) \mathrm{d}t \geq \beta, \quad \forall T \geq 0,\ \forall u \in \mathcal{L}_2. \tag{5}$$

If there are constants $\delta \geq 0$ and $\epsilon \geq 0$ such that

$$\int_0^T u^\top(t) y(t) \mathrm{d}t \geq \beta + \delta \int_0^T u^\top(t) u(t) \mathrm{d}t + \epsilon \int_0^T y^\top(t) y(t) \mathrm{d}t, \tag{6}$$

for all functions u, and all $T \geq 0$, then the system is input strictly passive (ISP) if $\delta > 0$, output strictly passive (OSP) if $\epsilon > 0$, and very strictly passive (VSP) if $\delta > 0$ and $\epsilon > 0$. In particular, for an LTI system with transfer function $H(s) = C(sI - A)^{-1}B + D$, with A Hurwitz and the pair (A, B) controllable it holds that [19]

1. The system is passive if and only if $\mathbf{Re}[H(j\omega)] \geq 0$ for all ω.
2. The system is ISP if and only if there is a $\delta > 0$ such that $\mathbf{Re}[H(j\omega)] \geq \delta > 0$ for all ω.
3. The system is OSP if and only if there is an ϵ such that $\mathbf{Re}[H(j\omega)] \geq \epsilon |H(j\omega)|^2$ for all ω.

Note that these three properties are checkable in a Nyquist diagram: if $H(j\omega)$ is in the closed right half plane then the system is passive, if $H(j\omega)$ is in $\mathbf{Re}[H(j\omega)] \geq \delta > 0$ the system is ISP and if $H(j\omega)$ is inside the circle with center in $s = \frac{1}{2\epsilon}$, and radius $r = \frac{1}{2\epsilon}$ then the system is OSP.

When the system is in state-space representation, the passivity of the input–output maps as before is replaced by the following notions from dissipative systems theory. Consider the system $H : \mathcal{L}_{2,e} \to \mathcal{L}_{2,e}$, given by

$$H : \begin{cases} \dot{x} = f(x, u), & u \in \mathbb{R} \\ y = g(x, u), & y \in \mathbb{R} \end{cases} \tag{7}$$

where f and g have the same properties as in (2). H is said to be *dissipative* with respect to a supply rate $w(u, y)$ if there exists a *storage function* $V(x) \geq 0$ such that the following dissipation inequality holds

$$V(x(T)) \leq V(x(0)) + \int_0^T w(u(t), y(t))\mathrm{d}t; \quad \forall u,\ \forall x(0),\ \forall T \geq 0. \tag{8}$$

Following [12], the relationship between dissipative and passive systems is given by the choice of a particular supply rate:

1. The system H is passive if it is dissipative with respect to supply rate $w_p = u^\top y$.
2. The system H is input strictly passive (ISP) if it is dissipative with respect to supply rate $w_i = u^\top y - \epsilon u^\top u$, for some $\epsilon > 0$.
3. The system H is output strictly passive (OSP) if it is dissipative with respect to supply rate $w_o = u^\top y - \delta y^\top y$, for some $\delta > 0$.
4. The system H is very strictly passive (VSP) if it is dissipative with respect to supply rate $w_v = u^\top y - \delta u^\top u - \epsilon y^\top y$, for some $\epsilon > 0$ and $\delta > 0$.

From (5) and (8) it is clear that β in the definition of a passive input–output map is related to the value of the storage function at the initial state by $\beta = -V(x(0))$.

An important result is the *Passivity Theorem* [12]: the feedback interconnection of Fig. 1 is finite-gain stable if

$$\epsilon_R + \delta_P > 0$$
$$\epsilon_P + \delta_R > 0$$

where ϵ_R and δ_R, and ϵ_P and δ_P are such that they satisfy (6) for the systems R and P, respectively.

3. Main results

The main goal of this work will be to analyze the passivity properties of reset compensators as described by (4). Firstly, conditions on the reset compensators are derived for them to be (strictly) passive systems. We have split the treatment of full reset and partial reset compensators into separate sections. Finally, the passivity theorem will be used to show stability properties of the feedback system given in Fig. 1. Notice that we allow for nonlinear plants as described by (2) as far as they are passive.

3.1. Full reset compensators

In spite of the fact that passivity theory (as an input–output theory) is not appropriate in general for hybrid systems, it can be successfully used for full reset compensators given by (4). This is due to the fact that the reset compensator looses all its memory at every reset time, i.e., the base compensator is restarted with zero initial condition. Therefore, a state-space description is not necessary because the reset compensator just depends on the input since the last reset time. This is the key point in the next proposition.

Proposition 1. *A full reset compensator R (given by (4) with $A_\rho = 0$) is passive, ISP, OSP, or VSP if the base compensator is passive, ISP, OSP, or VSP, respectively.*

Proof. The result is proved for the VSP case, the rest of the cases follow similar arguments. If the base compensator is VSP, then there exist $\delta > 0$ and $\epsilon > 0$ such that (6) is satisfied for all $T \geq 0$ and for all $u \in \mathcal{L}_2$. Since, by time regularization, in every finite time interval there are a finite number of resets instants $\{0 < t_1, t_2, \ldots\}$ with $0 < t_1 < t_2 < \cdots$, then there exists a finite integer k such that for every $T \in (t_k, t_{k+1}]$, the integration interval $[0, T]$ can be divided into a finite number of subintervals

$$\int_0^T u^\top(t)y(t)\mathrm{d}t = \int_0^{t_1} u^\top(t)y(t)\mathrm{d}t + \sum_{i=1}^{i=k-1} \int_{t_i}^{t_{i+1}} u^\top(t)y(t)\mathrm{d}t + \int_{t_k}^T u^\top(t)y(t)\mathrm{d}t. \tag{9}$$

In addition, since the base system is time-invariant and VSP by assumption, it is true that

$$\int_0^{t_1} u^\top(t)y(t)\mathrm{d}t \geq \delta \int_0^{t_1} u^\top(t)u(t)\mathrm{d}t + \epsilon \int_0^{t_1} y^\top(t)y(t)\mathrm{d}t, \tag{10}$$

$$\int_{t_i}^{t_{i+1}} u^\top(t)y(t)\mathrm{d}t \geq \delta \int_{t_i}^{t_{i+1}} u^\top(t)u(t)\mathrm{d}t + \epsilon \int_{t_i}^{t_{i+1}} y^\top(t)y(t)\mathrm{d}t \tag{11}$$

and

$$\int_{t_k}^T u^\top(t)y(t)\mathrm{d}t \geq \delta \int_{t_k}^T u^\top(t)u(t)\mathrm{d}t + \epsilon \int_{t_k}^T y^\top(t)y(t)\mathrm{d}t \tag{12}$$

for any $u \in \mathcal{L}_2$, and for $i = 1, 2, \ldots, k-1$. In (10)–(12), $\beta = 0$ has been taken due to the system has null initial conditions at t_i^+ for all $i = 1, 2, \ldots, k$. In addition, the integral limits t_i^+ can be taken as t_i by integral properties. Then, after direct substitution of (10)–(11)–(12) in (9) we directly conclude that

$$\int_0^T u^\top(t)y(t)\mathrm{d}t \geq \delta \int_0^T u^\top(t)u(t)\mathrm{d}t + \epsilon \int_0^T y^\top(t)y(t)\mathrm{d}t. \tag{13}$$

Thus, the reset compensator is a VSP system with the same constants $\beta(=0)$, δ and ϵ as its LTI base compensator. □

Remark 1.1. Note that formulation of passivity, ISP, OSP and VSP given in [12] (Section 2.2 and 3.1) is completely general, and thus does not involve any assumption regarding the continuity of system trajectories of the reset system R as given by (4). In particular, the state trajectories may be discontinuous at the reset instants, and application of passivity definitions and the passivity theorem can be directly applied to reset systems in order to get input–output stability properties.

Remark 1.2. Note that in the proof of Proposition 1, linearity of the base compensator has been used simply to state that it has null initial conditions at t_i^+ for all $i = 1, 2, \ldots$. Thus, Proposition 1 holds for any full reset compensator, not necessarily linear, as long as $\beta = 0$ or, using the dissipativity framework to be used in next Section, the storage function of the compensator satisfies $V(0) = 0$.

Some examples of reset compensators are analyzed in the following.

3.1.1. The first-order reset element (FORE)

A typical reset compensator is the *first-order reset element* (FORE) introduced in [2,3]. The transfer function of its base compensator is

$$FORE(s) = \frac{k}{s+b} \tag{14}$$

where it will be assumed that $k, b > 0$. The real part of its frequency response is given by

$$\mathbf{Re}[FORE(j\omega)] = \frac{kb}{b^2+\omega^2} \tag{15}$$

and thus, applying Proposition 1, it is directly obtained that

- *FORE* is passive since $\mathbf{Re}[FORE(j\omega)] > 0$ for all ω,
- *FORE* is OSP, since for $\epsilon = \frac{b}{k}$ it is satisfied that:

$$\mathbf{Re}[FORE(j\omega)] = \frac{b}{k}|FORE(j\omega)|^2. \tag{16}$$

3.1.2. A reset lag compensator (reset-PI)

In order to obtain VSP reset compensators, it is necessary that its base compensator has the same number of poles and zeros. A possible choice is a base lag compensator

$$PI(s) = k\frac{1+Ts}{1+\gamma Ts} \tag{17}$$

where $\gamma \in (1, \infty)$. Its frequency response real part is given by

$$\mathbf{Re}[PI(j\omega)] = k\frac{1+\gamma(T\omega)^2}{1+(\gamma T\omega)^2}, \tag{18}$$

then it is easy to check that $\mathbf{Re}[PI(j\omega)] > \min(1, \frac{k}{\gamma})$, and finally

$$|PI(j\omega)|^2 = k^2\frac{1+(T\omega)^2}{1+(\gamma T\omega)^2}. \tag{19}$$

Applying Proposition 1 it is now obtained that

- Reset-*PI* is passive due to $\mathbf{Re}[PI(j\omega)] > 0$ for all w,
- Reset-*PI* is ISP with $\delta = \min(1, \frac{k}{\gamma})$,
- Reset-*PI* is OSP with $\epsilon = \frac{\gamma}{k}$.

3.1.3. A reset-PID compensator

As example of higher-order reset element, consider as base a PID compensator with the transfer function:

$$PID(s) = k_p\gamma\frac{1+T_is}{1+\gamma T_is}\frac{1+T_ds}{1+\alpha T_ds} \tag{20}$$

where $0 \le T_d < T_i$, $1 \le \beta < \infty$, and $0 < \alpha \le 1$. This base compensator is VSP, hence the full reset compensator based on it, referred to as reset-PID, is VSP. Several other cases are possible for particular combination of parameters, a particular study of the Nyquist plot is necessary in each case.

3.2. Partial reset compensators

In the partial reset compensator case, that is the case in which only some of the compensator states are reset, dissipativity theory needs to be used (in contrast with the full reset case). Note that after each reset action, partial reset results in a possibly nonzero initial condition for the compensator state in comparison (as opposed to full reset, which produces a zero initial condition). Thus, for every initial condition after reset, a different input–output system has to be taken into consideration. As a result, the compensator state has to be included in the formulation and the result of Proposition 1 is not applicable. Dissipative systems theory leads to a solution of this problem due to the fact that the dissipativity property holds for every initial condition.

Proposition 2. *A partial reset compensator R (given by (4) with $A_\rho = diag(I_{n_{\bar\rho}}, 0_{n_\rho})$) is passive, ISP, OSP, or VSP if its base compensator is dissipative with respect to supply rate w_p, w_i, w_o, or w_v, respectively, and with a storage function $V(x)$ that satisfies*

$$V(A_\rho x) \le V(x) \tag{21}$$

for every $x \in \mathbb{R}^{n_r}$.

Proof. Again the case VSP is considered, the rest of the cases follow similar arguments. Since, by assumption, the base compensator is dissipative with respect to the storage function V and the supply rate $w_v = u^\top y - \delta u^\top u - \epsilon y^\top y$ for some $\delta > 0$ and $\epsilon > 0$, the following inequality is satisfied

$$V(x(T)) \le V(x(0)) + \int_0^T w_v(u(t), y(t))\mathrm{d}t,$$
$$\forall u \in \mathcal{L}_{2,e},\ \forall x(0) \in R^{n_r},\ \forall T \ge 0. \tag{22}$$

For a given input $u(t)$, by time regularization there is a finite number of reset times t_i, $i = 1, 2, \ldots, k$ in the interval [0,T], being $T \in (t_k, t_{k+1}]$. Thus, it is satisfied that

$$V(x(T)) \le V(x(t_k^+)) + \int_{t_k}^T w_v(u(t), y(t))\mathrm{d}t \tag{23}$$

since in the interval $[t_k^+, T]$ the reset compensator equals its base compensator with initial condition $x(t_k^+)$. Applying a similar argument it is true that

$$V(x(t_i)) \le V(x(t_{i-1}^+)) + \int_{t_{i-1}}^{t_i} w_v(u(t), y(t))\mathrm{d}t \tag{24}$$

for $i = 2, \ldots, k$, and also

$$V(x(t_1)) \le V(x(0)) + \int_0^{t_i} w_v(u(t), y(t))\mathrm{d}t. \tag{25}$$

Then, applying condition (21) at the reset time instants one obtains $V(x(t_i^+)) = V(A_\rho x(t_i)) \le V(x(t_i))$, $i = 1, \ldots, k$. Using this fact and (23)–(25), the final result is

$$\begin{aligned} V(x(T)) &\le V(x(t_{k-1}^+)) + \int_{t_{k-1}}^{t_k} w_v(u(t), y(t))\mathrm{d}t \\ &\quad + \int_{t_k}^{T} w_v(u(t), y(t))\mathrm{d}t \\ &= V(x(t_{k-1}^+)) + \int_{t_{k-1}}^{T} w_v(u(t), y(t))\mathrm{d}t \\ &\le V(x(t_{k-2}^+)) + \int_{t_{k-2}}^{T} w_v(u(t), y(t))\mathrm{d}t \\ &\le \cdots \end{aligned} \tag{26}$$

from which it is concluded that the reset compensator satisfies

$$0 \le V(x(T)) \le V(x(0)) + \int_0^T w_v(u(t), y(t))\mathrm{d}t. \tag{27}$$

As a result it is true that

$$\int_0^T u^\top(t)y(t)\mathrm{d}t \ge \beta + \delta\int_0^T u^\top(t)u(t)\mathrm{d}t + \epsilon\int_0^T y^\top(t)y(t)\mathrm{d}t, \tag{28}$$

where $\beta = -V(x(0))$, and thus the reset compensator is VSP. $\square$

Remark 2.1. A necessary and sufficient condition for the satisfaction of (21) can be given as follows for a convex storage function V. Note that if V is convex then V satisfies (21) for all $x \in \mathbb{R}^{n_r}$ if and only if $\frac{\partial V}{\partial x}(A_\rho x) \perp \ker A_\rho$ [20]. In the linear case (quadratic V) and A_ρ being given as the projection on the first vector component this is equivalent to the decoupled quadratic function V given by:

$$V(x) = x^\top \begin{bmatrix} Q_{11} & 0 \\ 0 & Q_{22} \end{bmatrix} x \tag{29}$$

where Q_{11} and Q_{22} are positive definite matrices with dimension $Q_{11} \in \mathbb{R}^{n_{\bar\rho} \times n_{\bar\rho}}$ and $Q_{22} \in \mathbb{R}^{n_\rho \times n_\rho}$. It is clear that V satisfies (21), since $V(A_\rho x) - V(x) = -x^\top \begin{bmatrix} 0 & 0 \\ 0 & Q_{22} \end{bmatrix} x \le 0, \forall x \in \mathbb{R}^{n_r}$.

Remark 2.2. A well-known condition for a LTI system with state-space representation (A, B, C, D) to be SPR is that

$$\begin{bmatrix} -QA - A^\top Q & QB - C^\top \\ B^\top Q - C & D + D^\top \end{bmatrix} > 0 \tag{30}$$

for some $Q > 0$. In addition, the system (A, B, C, D) is dissipative with respect to the storage function $V(x) = x^\top Qx$ [19]. Note that if the LMI (30) is satisfied for a block diagonal $Q = \begin{bmatrix} Q_{11} & 0 \\ 0 & Q_{22} \end{bmatrix}$ with the structure given by (29), then Proposition 2 can be applied and as a result the partial reset compensator with base system (A, B, C, D) will be VSP.

Some examples of passive partial reset compensators are given in the following.

3.2.1. Partial reset-PII_R compensator

Consider the high-order reset elements with base compensator PII_R defined as

$$PII_R(s) = k\frac{1 + \gamma_1 T_i s}{1 + T_i s}\frac{1 + \gamma_2 T_r s}{1 + T_r s} \tag{31}$$

where $\gamma_1 \in (0, 1)$ and $\gamma_2 \in (0, 1)$. Now, the transfer function is not enough to define the reset compensator because, with the same T_i, T_r, γ_1, and γ_2, an infinite amount of reset compensators can be defined. Thus, a space state description has to be given. In particular, consider the base compensator

$$PII_R(s) = \frac{1 + 2s}{1 + 3s}\frac{1 + 0.012s}{1 + s} \tag{32}$$

given by the block diagram of Fig. 2, where the left block gives the state $x_1 = u_1$ to be reset.

Then, the compensator is represented in state-space form as

$$A_r = \begin{bmatrix} -0.333 & 0.247 \\ 0 & -1 \end{bmatrix} \qquad B_r = \begin{bmatrix} 0.003 \\ 1 \end{bmatrix}$$

$$C_r = \begin{bmatrix} 0.444 & 0.658 \end{bmatrix} \qquad D_r = 0.008 \tag{33}$$

$$A_\rho = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}.$$

Note that by simply checking the Nyquist plot of the system (A_r, B_r, C_r, D_r) it is straightforward to show that it is SPR (Fig. 3). In addition, a decoupled storage function has been found which satisfies (29). It is given by

$$V(x) = x^\top \begin{bmatrix} 1.20 & 0 \\ 0 & 8.94 \end{bmatrix} x. \tag{34}$$

As a result, application of Proposition 2 (see Remark 2.2) guarantees that the partial reset PII_R compensator is VSP.

3.3. $\mathcal{L}_2$-stability of the reset control system

In this section, $\mathcal{L}_2$-stability will be studied for the feedback system of Fig. 1, by directly using Propositions 1 and 2, and the passivity theorem given in Section 2. Both the full reset and the partial reset cases will be considered.

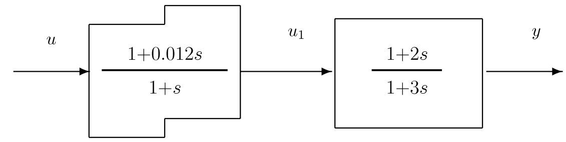

Fig. 2. Reset choice in $PII_R(s)$.

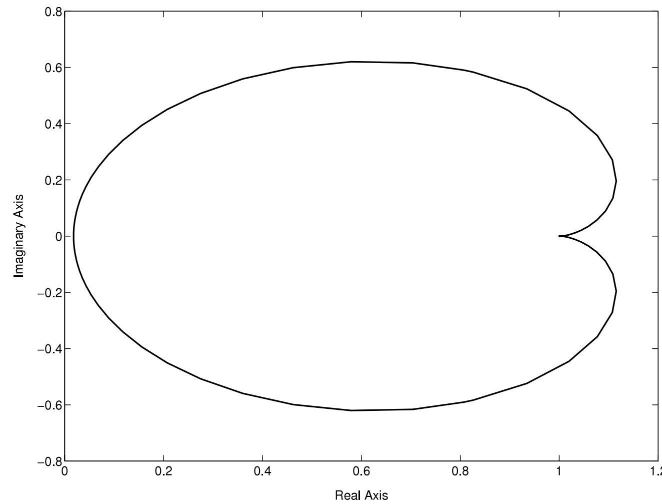

Fig. 3. Nyquist plot of $PII_R(s)$.

Proposition 3. *The reset feedback control system of the Fig. 1, with R being a full reset compensator with base compensator R_{bl}, is finite-gain stable if some of the following conditions are satisfied*

- *R_{bl} is ISP and P is ISP.*
- *R_{bl} is OSP and P is OSP*
- *R_{bl} is VSP and P is passive.*
- *R_{bl} is passive and P is VSP.*

In contrast to the full reset case, for the partial reset compensator case a state-space description is necessary. In this case, as it was shown in Proposition 2, a condition on the storage function has to be added.

Proposition 4. *The reset feedback control system of the Fig. 1, with R being a partial reset compensator with base linear compensator R_{bl}, is finite-gain $\mathcal{L}_2$ stable if some of the following conditions are satisfied*

- *R_{bl} is dissipative with respect to the supply rate w_i and a storage function V that satisfies (21), and P is ISP.*
- *R_{bl} is dissipative with respect to the supply rate w_o and a storage function that satisfies (21), and P is OSP.*
- *R_{bl} is dissipative with respect to the supply rate w_v and a storage function that satisfies (21) and P is passive.*
- *R_{bl} is dissipative with respect to the supply rate w_p and a storage function holds (21) and P is VSP.*

Note that if G is OSP (with parameters $\delta_G = 0$ and $\epsilon_G > 0$ as given by (6)), then G is finite L_2 gain stable with gain $\le \frac{1}{\epsilon_G}$ (see for example theorem 2.2.14 in [12]). Let us apply this to Propositions 3 and 4, for example in the last case where P is VSP (with parameters ϵ_P and δ_P in (6)), and R_{bl} is passive (and thus $\delta_R = \epsilon_R = 0$). Then, after simple manipulations we get that the closed-loop system (with inputs w, d, and outputs u, y) is finite L_2-gain stable with gain less than or equal to $\frac{1}{\min\{\epsilon_P, \delta_P\}} = \max\{\frac{1}{\epsilon_P}, \frac{1}{\delta_P}\}$. The other cases of Propositions 3 and 4 are similar. As a result, the gain of the reset control system is less than or equal to the gain of the base control system (while the precise computation of the gain of the closed-loop gain should be performed by other means).

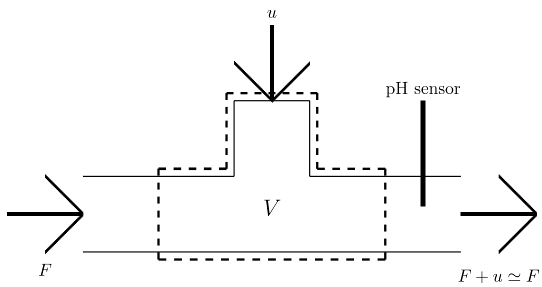

Fig. 4. In-line pH process.

4. Application to in-line pH control system

A reset compensator for an in-line pH control will be designed in this Section. Consider a process stream (with flow rate F) of distiller water which is mixed with a titrating stream (with flow rate u) of a strong acid in a continuously stirred tank reactor (CSTR) [21], with the following considerations (Fig. 4): (a) the mixture is perfect and instantaneous, (b) the pH sensor is placed very near to the acid injection, and (c) the flow rate of the titrating stream is negligible with respect to the flow rate of process stream, i.e., $F + u \simeq F$. Under these restrictions, the process can be modelled as a linear plant in cascade with a nonlinear passive memoryless system (Fig. 5).

The tank will be considered as the part of the pipe where acid is injected. Due to the conservation of charge, the sum of the concentration of the charges (W) is taken as state of the linear subsystem, i.e., $W = [H^+] - [OH^-]$. The dynamics of the CSTR is given by:

$$V\dot{W} = -F(W - W_{dw}) + uW_a \tag{35}$$

where V is the volume of the tank, F is the flow rate of process stream, u is the flow rate of the titrating stream and W_a is the concentration of charge in the titrating stream. In particular, for distilled water $W_{dw} = 0$, but also a stream process of strong base ($W_b < 0$) can be studied if $-W_b \ll W_a$. Furthermore, the output of the linear system equals the state, $y_L = W$. The relationship between this concentration and the pH is given by

$$pH(W) = -\log\left(\frac{W + \sqrt{W^2 + 4\ 10^{-14}}}{2}\right). \tag{36}$$

Taking an operation point, with a flow rate constant u_{ss}, an equilibrium point of (35) is obtained as $-FW_{ss} + u_{ss}W_a = 0$. Therefore, $W_{ss} = \frac{u_{ss}}{F}W_a$. Working with increments with respect to the equilibrium point, $\Delta W = W - W_{ss}$, the new dynamics is given by:

$$V\Delta\dot{W} = -F\Delta W + \Delta u W_a \tag{37}$$

where $\Delta u = u - u_{ss}$.

The nonlinear memoryless function is defined by $\Delta pH = pH(W_{ss}) - pH(W)$. The system was inverted in order that an increment in the input results in an increment in the output. Finally, the model is recasted as the block interconnection of Fig. 5, where L is a LTI system given by:

$$L : \begin{cases} \dot{x} = -ax + bu, \\ y = x, \end{cases} \tag{38}$$

where $a = \frac{F}{V}$ and $b = \frac{W_a}{V}$. The static nonlinearity is given

$$\phi(x) = pH(W_{ss}) - pH(x) \tag{39}$$

from which it follows that $x\phi(x) > 0$ for all $x \neq 0$. It can be easily shown (see for example [22] or [19]) that the system of Fig. 5 with L and ϕ given by (38)–(39) is passive.

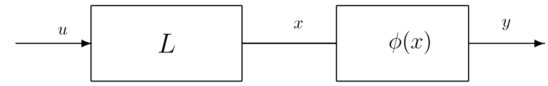

Fig. 5. Final model of the pH process under some considerations.

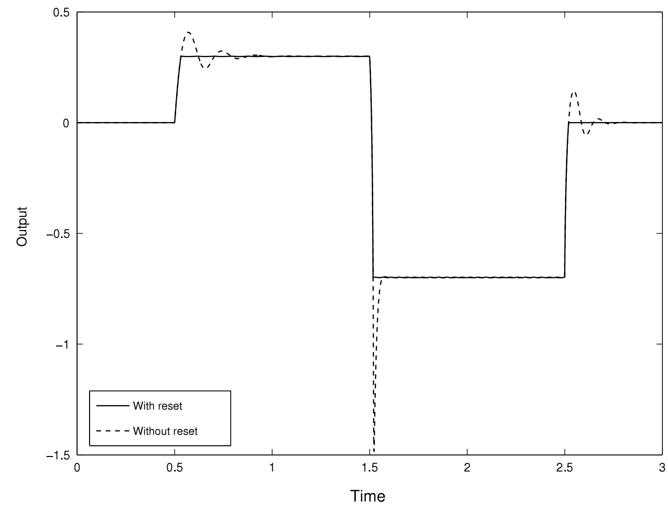

Fig. 6. Simulation of the pH process with a partial reset compensator.

A numerical example is developed for the following data: $V = 0.1$ l, $F = 5\ 10^{-2}$ l/s, $W_a = 1$ mol/l, $u_{ss} = 0.5\ 10^{-4}$ l/s. The linear plant from (38)

$$L(s) = \frac{20}{2s + 1} \tag{40}$$

and the static nonlinearity is given by

$$\phi(x) = 3 - \left(-\log_{10}\left(\frac{x + \sqrt{x^2 + 4\ 10^{-14}}}{2}\right)\right). \tag{41}$$

A reset-PII_R compensator has been also designed for tracking some pH references, consisting in deviations from the equilibrium point. By using Proposition 4, finite-gain $\mathcal{L}_2$-stability of the feedback system can be directly guaranteed since reset-PII_R is VSP. A simulation of the control system is shown in Fig. 6, where in addition an important characteristic of reset compensation is clearly shown: the possibility of obtaining very fast responses without overshoot, overcoming fundamental limits of LTI compensation.

5. Conclusions

In this work, a passivity approach has been used in order to guarantee the stability of a feedback interconnection between a reset compensator and a passive plant, including passive nonlinear plants. At the same time, passivity conditions on the reset compensators have been found in order to be used in passive control techniques. In the case of full reset compensators, it has been shown that passivity of the base compensator is sufficient, whereas in the case of partial reset compensation, an extra condition is needed: the storage function must be non-increasing at the reset instants. Some examples of passive reset compensators have been given, and in addition an in-line pH control problem has been solved and checked via simulation.

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